Kyle (1985)

Setup

1. The ex post value of the risky asset is

$$\tilde{v} \sim N(p_0, \Sigma_0)$$

The quantity traded by noise traders is

$$\tilde{u} \sim N(0, \sigma_u^2)$$

The random variables $\tilde{v}$ and $\tilde{u}$ are independently distributed. The quantity traded by the insider is denoted $\tilde{x}$ and the price is denoted $\tilde{p}$.

2. Trading is structured in two steps as follows:

   (a) The exogenous values of $\tilde{v}$ and $\tilde{u}$ are realised and the insider chooses the quantity she trades $\tilde{x}$. When doing so she observes $\tilde{v}$ but not $\tilde{u}$. The insider’s trading strategy, $X$, is a measurable function such that $\tilde{x} = X(\tilde{v})$ is justified.

   (b) The market makers determine the price $\tilde{p}$ at which they trade the quantity necessary to clear the market. When doing so they observe $\tilde{y} = \tilde{x} + \tilde{u}$ but not $\tilde{x}$ or $\tilde{u}$ (or $\tilde{v}$) separately. Their pricing rule, denoted $P$, is a measurable function such that $\tilde{p} = P(\tilde{y})$ is justified.

3. The profits of the insider are

$$\tilde{\pi} = (\tilde{v} - \tilde{p})\tilde{x}$$

To emphasise the dependence of $\tilde{\pi}$ and $\tilde{p}$ on $X$ and $P$, we write $\tilde{\pi} = \tilde{\pi}(X, P)$ and $\tilde{p} = \tilde{p}(X, P)$.

4. An equilibrium is defined as a pair $\{X, P\}$ such that the following conditions hold:

   (a) Profit maximisation. For any alternate trading strategy $X'$ and for any $v$,

   $$E[\tilde{\pi}(X, P)|\tilde{v} = v] \geq E[\tilde{\pi}(X', P)|\tilde{v} = v].$$

   (b) Market efficiency. The random variable $\tilde{p}$ satisfies

   $$\tilde{p}(X, P) = E[\tilde{v}|\tilde{y} = y].$$

Equilibrium

We wish to show that there exists a unique equilibrium in which $X$ and $P$ are linear functions.

Suppose that for constants $\mu$, $\lambda$, $\alpha$ and $\beta$, linear functions $P$ and $X$ are given by

$$P(y) = \mu + \lambda y \quad (1)$$

$$X(v) = \alpha + \beta v. \quad (2)$$

Then the insider’s expected profits conditional on the ex post value of the asset is

$$E[\tilde{\pi}|tv = v] = E[(v - P(\tilde{y}))x|\tilde{v} = v] = E[(v - \mu - \lambda \tilde{y})x|\tilde{v} = v] = (v - \mu - \lambda x)x, \quad (3)$$
since $E[\lambda u] = 0$. Hence, by the first equilibrium condition, the insider’s problem is to

$$\max_x (v - \mu - \lambda x)x,$$

which has FOC

$$0 = v - \mu - 2\lambda x$$

$$\Rightarrow X(v) = \frac{v - u}{2\lambda},$$

so we have

$$\beta = \frac{1}{2\lambda}$$

$$\alpha = -\mu\beta.$$  \hfill (6)

Thus, we have proved that in equilibrium, $X$ is a linear function.

Now, since $\tilde{y} = \tilde{x} + \tilde{u} = \alpha + \beta\tilde{v} + \tilde{u}$, it follows that

$$\tilde{y} \sim N(\alpha + \beta p_0, \beta^2 \Sigma_0 + \sigma_u^2),$$

so that

$$\left(\begin{array}{c} \tilde{v} \\ \tilde{y} \end{array}\right) \sim N\left(\left(\begin{array}{c} p_0 \\ \alpha + \beta p_0 \end{array}\right), \left(\begin{array}{cc} \Sigma_0 & \beta\Sigma_0 \\ \beta\Sigma_0 & \beta^2 \Sigma_0 + \sigma_u^2 \end{array}\right)\right)$$

since $\text{Cov}(\tilde{v}, \tilde{y}) = \beta\Sigma_0$.

By the market efficiency equilibrium condition,

$$p = E[\tilde{v} | \tilde{y} = y]$$

$$\mu + \lambda y = E[\tilde{v} | \alpha + \beta\tilde{v} + \tilde{u} = y].$$

An application of the conditional normal formula, given the distributional structure in (8), illustrates that

$$\lambda = \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2},$$

$$\mu = p_0 - \lambda(\alpha + \beta p_0).$$

Substitute (6) into (11), and solve for $\beta$ to get

$$\frac{1}{2\beta} = \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2}$$

$$\Rightarrow \beta^2 \Sigma_0 + \sigma_u^2 = 2\beta^2 \Sigma_0$$

$$\Rightarrow \beta^2 \Sigma_0 = \sigma_u^2$$

$$\therefore \beta = \left(\frac{\sigma_u^2}{\Sigma_0}\right)^{1/2}$$

$$\therefore \lambda = \frac{1}{2} \left(\frac{\Sigma_0}{\sigma_u^2}\right)^{1/2}.$$  \hfill (13)

Finally, substitute (7) into (12), so that

$$\mu = p_0 - \lambda(-\mu\beta + \beta p_0)$$

$$= p_0 + \lambda\beta(\mu - p_0)$$

$$\Rightarrow (1 - \lambda\beta)\mu = (1 - \lambda\beta)p_0$$

$$\therefore \mu = p_0$$

$$\therefore \alpha = -\beta p_0.$$  \hfill (15)
Hence, we have proved that there exists a unique equilibrium in which $X$ and $P$ are linear functions. Defining constants $\beta \triangleq (\sigma_u^2/\Sigma_0)^{1/2}$ and $\lambda \triangleq 2(\sigma_u^2/\Sigma_0)^{-1/2}$, the equilibrium $P$ and $X$ are given by

\begin{align*}
X(\tilde{v}) &= \beta(\tilde{v} - p_0) \\
P(\tilde{y}) &= p_0 + \lambda \tilde{y}.
\end{align*}

\begin{align*}
E[\pi] &= \frac{1}{2} \Sigma_0 \sigma_u^2 \\
V ar(\tilde{v}|P) &= \frac{\Sigma_0}{2}
\end{align*}