Reference

Admati (1985) is a multisecurity rational expectations model with noise. Investment decisions are based on mean-variance considerations, but each agent in effect uses a different model since they condition on different information. These conditional models do not naturally aggregate to imply similar unconditional models. Therefore, the market is generally not mean-variance efficient for any particular information set, including all public information. Uncertainty about the supply of one asset may prevent the prices of other assets from being fully revealing. This may represent a solution to the Grossman and Stiglitz (1980) paradox. The correlations among the assets can result in a number of strange results. Price may be decreasing in the profitability of an asset or increasing in its supply. The predicted payoff of an asset may be decreasing in price. Finally, assets may increase in price with greater demand.
Set-up

- a continuum of economic agents indexed by $a \in [0, 1]$,
- two periods; agents trade in period 0 and consume in period 1,
- each agent $a$ invests his initial wealth $W_{0a}$ between a riskless asset and $n$ risky assets,
- there is a single consumption good,
- the riskless asset pays $R$ units and risky assets pay $\tilde{F}$ units of the single consumption good, where $\tilde{F}$ is an $N \times 1$ vector,
- $\tilde{F} \sim MVN(\bar{F}, V)$,
- $P$ is the present price vector of risky assets (riskless asset is a numeraire),
- $D_a$ is a vector of holdings of the risky assets by agent $a$ (demand),
- agent $a$’s period 1 wealth is given by
  \[
  \tilde{W}_{1a} = (W_{0a} - D_a' P) R + D_a' \tilde{F}
  \]
  or, by rearranging
  \[
  \tilde{W}_{1a} = W_{0a} R + D_a' \left( \tilde{F} - RP \right)
  \] (1)
- agents have CARA utility functions
  \[
  u_a(W) = - \exp \left( - \frac{W}{\rho_a} \right)
  \] (2)
  where $(W|I) \sim N(\bar{W}, \sigma^2)$,
- each agent $a$ maximises $E_a(u_a(\tilde{W}_{1a}))$ where the expectation operator, $E_a$, is based on agent $a$’s information;
- agents are assumed to possess diverse and asymmetric pieces of private information, this information consists of the equilibrium price vector $P$ and the

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1A conceptual problem in a finite-agent model is that agents are assumed to be aware of the covariances between their error term and the price, and, at the same time, to be price takers. In the context of large economy the price taking assumption makes more sense, as these covariances can be ignored.
realisation of a **private information signal** $\tilde{Y}_a$ which is correlated with $\tilde{F}$, i.e.

a) $\tilde{Y}_a = \tilde{F} + \tilde{\epsilon}_a$, where $\tilde{Y}_a$, $\tilde{F}$ and $\tilde{\epsilon}_a$ are $N \times 1$ vectors,

b) $\tilde{\epsilon}_a$ and $\tilde{F}$ are independent,

c) $\tilde{\epsilon}_a \sim MVN (0, S_a)$ and $Cov[\tilde{\epsilon}_a, \tilde{\epsilon}_k] = 0$ for all $a \neq k$, i.e. agents private signals are uncorrelated and agents can vary in precision of their private information,

• the supply per capita of the risky assets is assumed to be the realisation of a random vector $\tilde{Z}$,

  a) for the first time the supply is stochastic!

  b) $\tilde{Z} \sim MVN (\bar{Z}, U)$ ,

  c) $\tilde{Z}$ is independent of $\tilde{\epsilon}_a$ and $\tilde{F}$.

• thus we have

  a) $\tilde{F} \sim MVN (\bar{F}, V)$ ,

  b) $\tilde{\epsilon}_a \sim MVN (0, S_a)$ ,

  c) $\tilde{Z} \sim MVN (\bar{Z}, U)$ ,

and each of $V$, $S_a$ and $U$ are positive definite,

• assume that each $\rho_a$ and all elements of $S_a^{-1}$ are uniformly bounded;

  a) this keeps the setting competitive: very risk tolerant or very well informed agents, if not ruled out, would appear "large," since they respond very sharply to their signals and might therefore affect the competitiveness of the economy,

  b) this boundedness assumption is used later in applying the law of large numbers,

• **average risk tolerance** is defined as

$$\bar{\rho} = \int_0^1 \rho_a da$$

i.e. we assume that agents can vary in their risk tolerance,
• **weighted average of the precision matrices**, where the weights are the risk tolerance coefficients, is defined as

\[
Q = \int_0^1 \rho_a S_a^{-1} da,
\]

• the economy is defined by a measurable function

\[
(\rho, S^{-1}) : [0, 1] \rightarrow \mathcal{R}_+ \times \mathcal{R}^{n \times n},
\]

where \((\rho_a, S_a^{-1})\), the value of this function at \(a \in [0, 1]\), is interpreted as the risk tolerance and the precision matrix of agent \(a\).

**Definition 1.** A rational expectations equilibrium (REE) is a price vector \(\hat{P}\) and allocation functions \(\{D_a(\hat{Y}_a, \hat{P})\}_{a \in [0, 1]}\) such that

(a) \(\hat{P}\) is \((\hat{F}, \hat{Z})\) measurable,

(b) optimisation under rational expectations:

\[
D_a(\hat{Y}_a, \hat{P}) \in \arg \max_{D_a} E \left[ u_a(\hat{W}_{1a}) | \hat{Y}_a, \hat{P} \right],
\]

(c) market clearing:

\[
\int_0^1 D_a(\hat{Y}_a, \hat{P}) da = \hat{Z}
\]

almost surely (a.s.).

Note that the model is quite rich, except for positive definiteness, no special assumptions are made concerning the variance-covariance matrices \(V, S_a\) and \(U\). Thus the model admits general correlation patterns across assets payoffs, supplies, and the error terms affecting each private signal. Asymmetry across the agents may arise both from differences in risk tolerance coefficients and from different patterns of private information, since the private signals are not assumed identically distributed. Agents might vary in the precision of their information about some or all the assets, and their error terms can be correlated across assets in a variety of ways. Indeed, we don’t even require that every \(S_a^{-1}\) be positive definite. All formulae hold in cases where some agents receive information only about a subset of assets, and
where many of them are completely uniformed. To ensure that there is "enough" information in the market, however, we assume that a positive measure of agents receive a proper $n$-dimensional signal, i.e., that for a positive measure of agents, $S_a^{-1}$ is positive definite.

**Demand functions**

First note that

$$E[u(w|I)] = -\exp\left\{-\frac{1}{\rho_a} \left[ \bar{w} - \frac{\sigma^2}{2\rho_a} \right] \right\}$$

since $W|I \sim N(\bar{W}, \sigma^2)$ and using the normal distribution moment generating function. Therefore maximising the expected utility is equivalent to maximising

$$\max_{D_a} \left\{ E[\tilde{W}_{1a}|I] - \frac{1}{2\rho_a} Var[\tilde{W}_{1a}|I] \right\}$$

(4)

As $\tilde{W}_{1a} = W_{0a}R + D'_a \left( \bar{F} - RP \right)$, we have

$$E[\tilde{W}_{1a}|I] = E\left[ W_{0a}R + D'_a \left( \bar{F} - RP \right) \right] = W_{0a}R + D'_a \left( E[\bar{F}|I] - RP \right)$$

$$Var[\tilde{W}_{1a}|I] = Var\left[ W_{0a}R + D'_a \left( \bar{F} - RP \right) \right] = D'_a Var[\bar{F}|I] D_a$$

therefore (4) becomes:

$$\max_{D_a} \left\{ W_{0a}R + D'_a \left( E[\bar{F}|I] - RP \right) - \frac{1}{2\rho_a} D'_a Var[\bar{F}|I] D_a \right\}$$

(5)

Differentiating (5) with respect to $D_a$ yields the following F.O.C.:

$$E[\bar{F}|I] - RP = \frac{1}{\rho_a} Var[\bar{F}|I] D_a$$

which after rearranging gives the demand function for an individual $a$

$$D_a = \rho_a Var[\bar{F}|I]^{-1} \left( E[\bar{F}|I] - RP \right)$$

(6)
Theorem 1 (Rational Expectations Equilibrium Price). There exists a unique rational expectations equilibrium price within the class of functions of the form

\[ P = A_0 + A_1 \tilde{F} - A_2 \tilde{Z} \quad (7) \]

with \( A_2 \) nonsingular. This price has

\[ A_0 = \frac{\tilde{\rho}}{R} \left( \tilde{\rho} V^{-1} + \tilde{\rho} Q U^{-1} Q + Q \right)^{-1} (V^{-1} \tilde{F} + Q U^{-1} \tilde{Z}) ; \quad (8) \]
\[ A_1 = \frac{1}{R} \left( \tilde{\rho} V^{-1} + \tilde{\rho} Q U^{-1} Q + Q \right)^{-1} (Q + \tilde{\rho} Q U^{-1} Q) ; \quad (9) \]
\[ A_2 = \frac{1}{R} \left( \tilde{\rho} V^{-1} + \tilde{\rho} Q U^{-1} Q + Q \right)^{-1} (I Q + \tilde{\rho} Q U^{-1}) . \quad (10) \]

Proof. To prove, we set demand equal to supply and solve for price. We start by noting that in equilibrium, the market clearing condition is \( \int_0^1 D_a(\tilde{Y}_a, \tilde{P}) da = \tilde{Z} \) a.s. Using (6), we can re-write it as

\[ \int_0^1 \rho_a \text{Var} \left[ \tilde{F} | \tilde{Y}_a, \tilde{P} \right]^{-1} \left( E \left[ \tilde{F} | \tilde{Y}_a, \tilde{P} \right] - RP \right) da = \tilde{Z} \text{ a.s.} \quad (11) \]

Now

\[
\begin{pmatrix}
\tilde{F} \\
\tilde{Y}_a \\
\tilde{P}
\end{pmatrix}
\sim
MVN \left\{ \begin{pmatrix}
\tilde{F} \\
\tilde{F} \\
A_0 + A_1 \tilde{F} - A_2 \tilde{Z}
\end{pmatrix}, \begin{pmatrix}
V & V & VA_1' \\
V & V + S_a & VA_1' \\
A_1 V & A_1 V & A_1 V A_1' + A_2 U A_2'
\end{pmatrix} \right\}
\]

Recall the conditional properties of the multivariate normal distribution:

\[ E[X_1|X_2] = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (X_2 - \mu_2) \]
\[ V ar[X_1|X_2] = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}. \]

Note that \( \Sigma_{12} \Sigma_{22}^{-1} \) is termed the matrix of regression coefficients of \( X_1 \) on \( X_2 \).

2The nonsingularity of both \( A_2 \) and \( U \) prevents linear rational expectations equilibrium prices from being a sufficient statistics for any combination of the private information signals. Indeed, each agent will use his/her entire vector of private information in addition to the equilibrium prices in making conditional assessments. This eliminates the conceptual problems associated with fully revealing prices.

3If a random vector \( X = [X_1, ..., X_N] \) follows multivariate normal distribution, i.e.

\[ X \sim MVN(\mu, \Sigma), \]
Hence
\[ E \left[ \bar{F} | \bar{Y}_a, \bar{P} \right] = B_{0a} + B_{1a} Y_a + B_{2a} P \]  \hspace{1cm} (12)

where
\[ \left( \begin{array}{cc} B_{1a} & B_{2a} \end{array} \right) = \left( \begin{array}{cc} V & VA' \\ V + S_a & VA' + A_2 U A'_1 \end{array} \right)^{-1} \]

\[ \Rightarrow \left( \begin{array}{cc} B_{1a} & B_{2a} \end{array} \right) \left( \begin{array}{cc} V + S_a & VA' \\ A_1 V & VA' + A_2 U A'_1 \end{array} \right) = \left( \begin{array}{cc} V & VA' \\ A_1 V & VA' + A_2 U A'_1 \end{array} \right) \]

\[ \Rightarrow B_{1a}(V + S_a) + B_{2a} A_1 V = V \] \hspace{1cm} (13)

Using the regression coefficients idea, we get
\[ V_a = Var \left[ \bar{F} | \bar{Y}_a, \bar{P} \right] = V - \left( \begin{array}{cc} B_{1a} & B_{2a} \end{array} \right) \left( \begin{array}{cc} V \\ A_1 V \end{array} \right) \]

\[ \Rightarrow V_a = V - B_{1a} V - B_{2a} A_1 V \] \hspace{1cm} (14)

Plugging (12) and (14) into (11) gives:
\[ \int_0^1 \rho_a V_a^{-1} [B_{0a} + B_{1a} Y_a + B_{2a} P - R P] \, da = \tilde{Z} \, a.s. \]

\[ \Rightarrow \int_0^1 \rho_a V_a^{-1} [B_{0a} + B_{1a} Y_a + (B_{2a} - R I) P] \, da = \tilde{Z} \, a.s. \]

and if \( \mu \) and \( \Sigma \) are partitioned in the following way:
\[ \mu = \left( \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right) \] with sizes \( (q \times 1) \times (N - q) \times 1 \)
\[ \Sigma = \left( \begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array} \right) \] with sizes \( (q \times q) \times (N - q) \times 1 \times (N - q) \times (N - q) \)

then the distribution of \( X_1 \) conditional on \( X_2 = a \) is multivariate normal \( X_1 | X_2 \sim MVN (\bar{\mu}, \bar{\Sigma}) \) where
\[ \bar{\mu} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (a - \mu_2) \]

and covariance matrix
\[ \bar{\Sigma} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}. \]

Note that \( \Sigma_{12} \Sigma_{22}^{-1} \) is termed the matrix of regression coefficients of \( X_1 \) on \( X_2 \).

4A full description of the conditional distribution of \( \left( \bar{F} | \bar{Y}_a, \bar{P} \right) \) is given in Corollary 3.3 in Admati (1985), p. 639.
where \( I \) is the \( n \)-dimensional identity matrix. This can be written as

\[
\int_0^1 \rho_a V_a^{-1} B_{0a} da + \int_0^1 \rho_a V_a^{-1} B_{1a} Y_a da + \int_0^1 \rho_a V_a^{-1} (B_{2a} - RI) P da = \tilde{Z} \quad \text{a.s.} \quad (15)
\]

Now (7) will be a rational expectations equilibrium price if and only if the following conditions, which are obtained by rewriting (15) and equating coefficients, hold:

\[
A_2^{-1} = \int_0^1 \rho_a V_a^{-1} (RI - B_{2a}) da, \quad (16)
\]

\[
A_1 = A_2 \int_0^1 \rho_a V_a^{-1} B_{1a} da, \quad (17)
\]

\[
A_0 = A_2 \int_0^1 \rho_a V_a^{-1} B_{0a} da, \quad (18)
\]

and if, in addition, it is justified to write

\[
\int_0^1 \rho_a V_a^{-1} B_{1a} \tilde{\varepsilon}_a da = 0 \quad \text{a.s.} \quad (19)
\]

We need (19) because we have

\[
A_2 \int_0^1 \rho_a V_a^{-1} B_{1a} Y_a da = A_2 \int_0^1 \rho_a V_a^{-1} B_{1a} \left( \tilde{F} + \tilde{\varepsilon}_a \right) da = A_2 \int_0^1 \rho_a V_a^{-1} B_{1a} \tilde{F} \, da + A_2 \int_0^1 \rho_a V_a^{-1} B_{1a} \tilde{\varepsilon}_a \, da
\]

Notice that \( \tilde{\varepsilon}_a \) has mean zero and a bounded variance. Using the strong law of large numbers (Chiang 5.5.4), we have \( \frac{1}{n} \sum_{i=1}^n \tilde{\varepsilon}_{ai} \to 0 \) a.s. Thus it seems natural to define \( \int_0^1 \tilde{\varepsilon}_a da = 0 \), which implies \( \int_0^1 \tilde{Y}_a da = \tilde{F} \) a.s.

Thus we have

\[
P = A_2 \int_0^1 \rho_a V_a^{-1} B_{0a} \, da + A_2 \int_0^1 \rho_a V_a^{-1} B_{1a} \tilde{F} \, da - A_2 \tilde{Z}
\]

\[
P = A_0 + A_1 \tilde{F} - A_2 \tilde{Z}
\]