AN EXPLICIT BOUND ON INDIVIDUAL ASSETS' DEVIATIONS FROM APT PRICING IN A FINITE ECONOMY*

Philip H. DYBVIG

Yale School of Management, New Haven, CT 06520, USA

Received December 1982, final version received May 1983

Ross's Arbitrage Pricing Theory (APT) is a tractible and reasonable alternative to the mean-variance model. Nonetheless, understanding of the theory has been obscured by the complexity of the sequence economy models used for motivation. By contrast, we give a simple and direct derivation of the APT in a finite economy. Using an explicit bound on the deviations from APT prices across assets, a coarse calculation shows that theoretical deviations from APT pricing are negligible in our economy.

1. Introduction

The Arbitrage Pricing Theory (APT) of Ross (1976a,b) has attracted considerable attention from theorists and empiricists alike, since it gives a tractible and reasonable alternative to the mean-variance model. In the past, rigorous theoretical justifications of the theory have been developed in models with sequences of economies, with convergence to APT pricing in the limit. [Notable exceptions are Grinblatt and Titman (1983), Titman (1982), and Ross (1982).] Consequently, the models have been relatively inaccessible, and the limiting results have left open the question of whether APT pricing should hold approximately in our economy. The purpose of this paper is to present a model which overcomes these objections. The model works with a fixed economy, so it is simpler and more accessible than the

*Special thanks for helpful suggestions go to Jon Ingersoll, Steve Ross, Greg Connor, and the referee, Dick Roll. Grinblatt and Titman (1983) [and an earlier draft by Titman (1982)] have independently derived many similar results. The author was a Batterymarch fellow while the research presented here was completed.

Grinblatt and Titman (1983) is very similar to this paper. The bound of Grinblatt and Titman's is based on a bound relating $u''(.)$ and $Eu'(.)$, while this paper has a bound based on the harmonic mean of absolute risk aversion coefficients. The other major difference is that their bound is based on an assumption that factors may be purchased, at least approximately, in a large economy. Instead, this paper assumes aggregation. Ross (1982) works in a diffusion model, and gets an exact bound similar to an exact bound that Grinblatt and Titman get in a section which assumes normality or quadratic utility. Titman (1982) is an early version of Grinblatt and Titman (1983).
limit economy models. Furthermore, the model gives an explicit bound on the error.

The seminal paper by Ross (1976b) gave the first formal derivation of the APT. Being first forgives some flaws in the paper including a non-intuitive proof and a complicated argument. Huberman (1982) illustrates a simple intuition relating an asymptotic no arbitrage condition to convergence to APT pricing. Huberman's paper is less ambitious than the original Ross paper, since it does not start from first principals (preferences and payoff distributions) and takes as given the asymptotic no arbitrage condition.

Connor (1980) has pointed out a weakness in Ross's original paper which also applies to Huberman's paper: the model implies only that assets will be priced correctly on average (in cross section), so that a few of many assets can be grossly mispriced, even as the economy grows. Connor points out that what is needed is a restriction on asset supplies to make sure that no idiosyncratic risk is large relative to the economy as a whole. His model embodies that observation. However, his model is complicated and has strong assumptions. In addition, his model is subject to a criticism similar to his criticism of Ross's original model. His approximation is valid only if there are sufficiently many assets each having sufficiently small supply relative to the economy. However, we do not know how many assets are sufficiently many or how small a relative supply is sufficiently small, so the approximation is not useful for a finite economy. (Is Exxon small enough to be priced correctly by the APT?) This criticism is valid as well for the more recent model of Ingersoll (1981), and for the model of Chamberlain and Rothschild (1983) who work directly in the limit.

The model in this paper works directly in a finite economy, starting with simple assumptions on preferences and the factor model. The deviations of asset prices from the APT prices are explicitly bounded by an expression in terms of quantities which are directly observable or easily estimable: per capita asset supplies, individual bounds on absolute risk aversion, variance of idiosyncratic risk, and the interest rate. Therefore, we can reasonably expect the model to tell us whether, for example, Exxon stock should be priced correctly by the APT.

The proof works in two parts. First, a bound is derived for an economy with a single agent. Then, for a multi-agent economy, Pareto optimality is assumed to ensure the existence of a representative agent. A bound on the representative-agent's absolute risk aversion, depending on individual agents' absolute risk aversion bounds, is then derived — the representative-agent's bound is the harmonic mean of the individuals' bounds (as is familiar from diffusion models). Using the single-agent bound on the representative agent gives the result. The result should be contrasted with other simplified proofs of the APT, which assume that some agent holds a portfolio which is approximately well-diversified. [See Cragg and Malkiel (1982), Dybvig (1981),
and Chen and Ingersoll (1983).] Those results effectively assume the noise is insignificant in pricing, whereas this paper puts a derived bound on the significance of idiosyncratic noise in determining asset prices.

Section 2 derives the APT for a single agent economy, with the explicit bound on the deviation from APT pricing. Section 3 derives the APT and the bound on the deviation for a multi-agent economy, and section 4 relates the result to a model with a sequence of economies. A rough calculation of the bound is done in section 5. This computation indicates that APT pricing should be a very good approximation in our economy. Section 6 closes the paper.

2. APT in the single-agent economy

The basic underlying assumption of the APT is that asset payouts are generated by a factor model. In other words, the payoff \( \tilde{x}_i \) of asset \( i \) is given by

\[
\tilde{x}_i = c_i + \sum_{j=1}^{J} \beta_{ij} \tilde{f}_j + \tilde{e}_i,
\]

where \( c_i \) is a constant, \( \tilde{f}_j \) is the value of the \( j \)th factor, \( \beta_{ij} \) is the loading of the \( j \)th factor in the \( i \)th asset, and \( \tilde{e}_i \) is asset \( i \)'s idiosyncratic noise term. (Tildes indicate random variables.) Later we will put more structure on the variables in (1), but for now assume that the idiosyncratic noise terms \( \tilde{e}_i \) are independent of the factors \( \tilde{f}_j \) and each other, and that \( \mathbb{E}(\tilde{e}_i) = 0 \) for all \( i \) and \( \mathbb{E}(\tilde{f}_j) = 0 \) for all \( j \).

Let \( \tilde{w} = \sum \alpha_i \tilde{x}_i \) be the terminal wealth of some agent with portfolio holding \( \alpha \) and von Neumann–Morgenstern utility function \( u(.) \) of terminal wealth. If this agent has initial wealth \( w_0 \) and faces asset prices \( p_i \), the maximization problem is of the following form.

**Problem 1.** Choose \( \alpha_1, \alpha_2, \ldots, \alpha_n \) to

\[
\max \mathbb{E}u\left( \sum_i \alpha_i \tilde{x}_i \right) \quad \text{subject to} \quad \sum_i p_i \alpha_i = w_0.
\]

The first-order condition to Problem 1 is that there exists \( \mu \) (the marginal utility of initial wealth) such that, for all assets \( i \),

\[
\mathbb{E}\left[ \tilde{x}_i u\left( \sum_i \alpha_i \tilde{x}_i \right) \right] = \mu p_i.
\]
Substituting in (1) for \( x_i \) and using the definition of \( \tilde{w} \),

\[
p_i = c_i \frac{E[u'(\tilde{w})]}{\mu} + \sum_{j=1}^{J} \beta_{ij} \frac{E[\tilde{f}_j u'(\tilde{w})]}{\mu} + \frac{E[\tilde{e}_i u'(\tilde{w})]}{\mu}.
\]

The conclusion of the APT is that the price of asset \( i \) should depend linearly (across assets) on its constant term and its factor loadings \( \beta_{ij} \), but not on its own idiosyncratic noise. In (3), this follows if the last term is small for all \( i \).\(^2\) This is the result we are after. Eq. 3 was first derived by Connor (1980), and Cragg and Malkiel (1982).

Intuitively, since \( \tilde{e}_i \) has mean zero and has a small contribution to aggregate variability, it is plausible to expect that the last term in (3) is small. However, we need more assumptions to prove that. In an economy with many agents, there may not be any agent who holds a portfolio with small \( \alpha_i \)'s even if aggregate supply has small proportions of each asset. Furthermore, even if there is only one agent in the economy, small risk by some measures (e.g., variance) may not be considered small by the agent.\(^3\)

We address these two problems separately. First, we look at a factor economy with a single agent, and give sufficient conditions for the deviation from APT pricing in (3) to be small. Afterwards, we give sufficient conditions for the single-agent result to generalize to a market with many agents.

The single-agent result requires assumptions that ensure that \( \tilde{e}_i \) risk is actually small. Two of our assumptions will combine to imply that \( \tilde{e}_i \) risk does little damage: an assumption that the agent's absolute risk aversion is bounded above and an assumption that \( \tilde{e}_i \) is never smaller than \(-1\). The assumption on \( \tilde{e}_i \) is also a normalization: so long as \( \tilde{e}_i \) is bounded below, the share size of asset \( i \) can be defined to make \( \tilde{e}_i \) bounded below by \(-1\). This is a very weak assumption which is true with a normalization, for example, if asset payoffs have limited liability (i.e., never go negative), given that \( \tilde{e}_i \) is independent of the \( \tilde{f}_j \)'s.\(^4\) The formal statement follows.

**Theorem 1.** Suppose that asset payoffs are given by the factor eq. (1) and that the single agent in the economy faces Problem 1 and, in equilibrium, chooses to hold the market portfolio \((\alpha_1, \ldots, \alpha_J)\). Let

\(^2\)The APT may hold even if these terms are large, but for different coefficients on \( c_i \) and the \( \beta_{ij} \)'s. We will be giving sufficient conditions for the APT and it is certainly sufficient for these terms to be small.

\(^3\)For an extreme case, consider an agent with log or other CRRA utility, who would find that addition to wealth of a normal random variable of arbitrarily small variance would make him infinitely worse off, due to the infinite disutility of income at negative wealth levels. Even for CARA (exponential) utility, small variance does not imply small damage. This can be seen by noting that \(-e^{-w}\) goes to \(-\infty\) faster than \(-w^2\), as \( w \rightarrow -\infty \).

\(^4\)If \( q_i \) is any positive value taken on by \( c_i + \sum_{j=1}^{J} \beta_{ij} f_j \), then let \( q_i \) new shares be the same as one old share. If there is no arbitrage, \( q_i \) can be chosen at least as small as \( p_i(1+r) \) where \( r \) is the interest rate, so per capita supply (which enters into the bound) can increase by no more than this factor at the very most.
(a) \( \mathbb{E}(\tilde{e}_i) = 0, \ \mathbb{E}(\tilde{f}_j) = 0, \) the \( \tilde{e}_i \)'s are independent of \( \tilde{f}_j \)'s and each other, and \( \Pr(\epsilon_i \geq -1) = 1. \)

(b) The agent has an increasing and strictly concave von Neumann-Morgenstern utility function \( u \) which is three times continuously differentiable, has absolute risk aversion \( A(w) = -u''(w)/u'(w) \) which is uniformly bounded above by \( \overline{A} \), and satisfies \( u'''(w) > 0. \)

(c) Each asset is in positive net supply, i.e., \( \alpha_i > 0 \) for all \( i. \) (Of course, we can ignore any asset in zero net supply in a single-agent economy.)

Then, for all \( i, \)

\[
p_i = \rho c_i + \sum_{j=1}^{J} \beta_{ij} \lambda_j + \delta_i, (4)
\]

where \( \rho \equiv \mathbb{E}u'(\tilde{w})/\mu \) is the implicit riskless discount factor, \( \lambda_j \equiv \mathbb{E}[\tilde{f}_j u'(\tilde{w})]/\mu \) is the discount (or premium) paid for assuming risk from factor \( j, \) and the deviation from APT pricing satisfies the explicit bound

\[
|\delta_i| \leq \rho e^{\lambda \alpha_i} \text{var}(\tilde{e}_i) \overline{A} \alpha_i. \quad (5)
\]

**Proof.** With the definitions of \( \rho \) and \( \lambda_j \) in the statement of the theorem, (4) is the same as (3), with

\[
\delta \equiv \mathbb{E}[\tilde{e}_i u'(\tilde{w})]/\mu = \rho \mathbb{E}[\tilde{e}_i u'(\tilde{w})]/\mathbb{E}[u'(\tilde{w})]. (6)
\]

Since

\[
\tilde{w} = \sum_{i=1}^{I} \alpha_i \tilde{x}_i = \sum_{i=1}^{I} \alpha_i \left(c_i + \sum_{j=1}^{J} \beta_{ij} \tilde{f}_j + \tilde{e}_i \right),
\]

where \( \tilde{e} \) is independent of all other random variables, we can write

\[
\tilde{w} = \tilde{w}^*_t + \alpha_i \tilde{e}_i,
\]

where \( \tilde{w}^*_t \) and \( \tilde{e}_i \) are independent. Rewriting (6), we have that

\[
\delta_i = \rho \mathbb{E}[\tilde{e}_i u'(w^*_t + \alpha_i \tilde{e}_i)]/\mathbb{E}[u'(w^*_t + \alpha_i \tilde{e}_i)]. (7)
\]

Since \( \mathbb{E}(\tilde{e}_i) = 0 \) and \( \tilde{e}_i \geq -1, \) a Taylor expansion\(^5\) gives that

\[\begin{align*}
\mathbb{E}[\tilde{e}_i u'(w^*_t + \alpha_i \tilde{e}_i)] &= \mathbb{E}[\tilde{e}_i u'(w^*_t + \alpha_i \tilde{e}_i)] \\
&= \mathbb{E}[\tilde{e}_i \tilde{u}'(\xi)] \text{ [since } \mathbb{E}(\tilde{e}_i) = 0] \\
&< \alpha_i \mathbb{E}[\mathbb{E}[\tilde{e}_i \tilde{u}'(\xi) | w^*_t]] \\
&\leq \alpha_i \mathbb{E}[\mathbb{E}[\tilde{e}_i \tilde{u}'(\xi) | w^*_t] - \mathbb{E}[\tilde{e}_i \tilde{u}'(\xi)]] \\
&\leq \alpha_i \mathbb{E}[\mathbb{E}[\tilde{e}_i \tilde{u}'(\xi) | w^*_t] - \mathbb{E}[\tilde{e}_i] \tilde{u}(\tilde{w})] \\
&\leq \alpha_i \mathbb{E}[\mathbb{E}[\tilde{e}_i \tilde{u}'(\xi) | w^*_t] - \mathbb{E}[\tilde{e}_i] \tilde{u}(\tilde{w})].
\end{align*}\]

which is as desired, since \( \mathbb{E}(\tilde{e}_i^2) = \text{var}(\tilde{e}_i). \) Note that, here and elsewhere, we use the exact form of Taylor's theorem [see Fleming (1965, pp. 311-312)].
Absolute risk aversion bounded above by $A$ implies\(^6\) that the sup in (8) is bounded above by $u'(w^*_i)\bar{A}e^{A\zeta}$. Furthermore, convexity of $u'$ (namely $u'' > 0$) and $E(\bar{e}_i) = 0$ imply that the denominator of (7),

$$E[u'(w^*_i + \bar{e}_i)] > E[u'(w^*_i)].$$

Taking (7), (8), the comment after (8), and (9) together, we get that

$$\left| \delta_i \right| \leq \rho \alpha_i E[u'(\bar{w}^*_i) e^{A\zeta} \text{var}(\bar{e}_i)] / E[u'(\bar{w}^*_i)]$$

$$= \rho e^{A\zeta} \text{var}(\bar{e}_i) \bar{A} \alpha_i,$$

since the $E[u'(w^*_i)]$ in the numerator cancels with the denominator. Q.E.D.

The bound in (5) tells us that the deviation from APT pricing will be small if $\bar{A} \alpha_i$ is small and if $\rho$ and the variance of $\bar{e}_i$ are not too large. Normally, the discount factor $\rho \leq 1$, corresponding to a non-negative interest rate, as occurs whenever free storage is possible. If $\bar{A}$ were large, the agent might care deeply about even small risks, and if $\alpha_i$ were large, the risk in asset $i$ might be large relative to the economy.

Theorem 1 has an assumption ($u'' > 0$) which we did not discuss earlier. This assumption is weaker than (i.e., implied by assuming) decreasing absolute risk aversion. In fact, this assumption is not needed if the bound is stated differently.\(^7\)

\(^6\)We want to show that

$$\sup \left\{ \left| u'(w) \right| w \geq w^*_i - \zeta \right\} \leq u'(w^*_i) \bar{A} e^{A\zeta}.$$

Absolute risk aversion bounded by $\bar{A}$ implies that

$$\sup \left\{ \left| u'(w) \right| w \geq w^*_i - \zeta \right\} \leq \bar{A} \sup \{ u'(w) | w \geq w^*_i - \zeta \} = \bar{A} u'(w^*_i - \zeta),$$

since $u'' < 0$. But,

$$\log(u'(w^*_i - \zeta)) = \log u'(w^*_i) - \int_{w^*_i - \zeta}^{w^*_i} \frac{u'(w)}{u'(w)} dw$$

[since $u'(w)/u'(w)$ is the derivative of $\log(u'(w))$]

$$\leq \log u'(w^*_i) + \int_{w^*_i - \zeta}^{w^*_i} \bar{A} dw = \log u'(w^*_i) + \alpha_i \bar{A}. \text{ so}$$

$$u'(w^*_i - \zeta) = e^{\log(u'(w^*_i) + \bar{A} \alpha_i) - \frac{\bar{A}}{\alpha_i}}.$$ Q.E.D.

\(^7\)As a preliminary, let us prove that $-u''/u' \leq \bar{A}$ implies that for all $\bar{A} > 0$, $u'(w + A) \geq u'(w)e^{-\bar{A} A}$. To see this, note that

$$\log u'(w + A) = \log u'(w) + \int_w^{w + A} \frac{d}{dw} (\log u'(w)) dw = \log u'(w) + \int_w^{w + A} \frac{u''(w)}{u'(w)} dw$$

$$= \log u'(w) - \int_w^{w + A} A(w) dw \geq \log u'(w) - \bar{A} A.$$
Since asset pricing tests are usually performed using returns rather than prices, it is useful to rewrite the bound of Theorem 1 in terms of expected returns. Specifically, the return $r_i$ to asset $i$ is, by (1),

$$\tilde{r}_i = \tilde{x}_i / p_i = c_i / p_i + \sum_{j=1}^{J} \beta_{ij} \tilde{f}_j / p_i + \tilde{\varepsilon}_i / p_i,$$

and

$$E(\tilde{r}_i) = c_i / p_i.$$ 

Therefore, rewriting (4) in terms of returns gives us

$$E(\tilde{r}_i) = \rho^{-1} - \sum_{j=1}^{J} \frac{\beta_{ij} \lambda_j}{p_i} - \rho^{-1} \frac{\delta_i}{p_i},$$

where $\beta_{ij} / p_i$ is a factor loading for returns, and $\rho^{-1}(\delta_i / p_i)$ is the deviation from the APT, in terms of expected return. The following result is immediate from (5) of Theorem 1.

**Corollary 1.** Under the conditions of Theorem 1, the bound on deviations from APT pricing, in terms of expected return, is given by

$$\left|\rho^{-1} \frac{\delta_i}{p_i}\right| \leq e^{\lambda_2} \text{var}(\tilde{\varepsilon}_i / p_i) A_2 p_i.$$

This bound is the same as in Theorem 1, except that the variance is now the variance of return due to idiosyncratic noise, and the supply $x_i$ in shares is replaced by supply $x_i p_i$ in value (except in the exponent).

We have completed our derivation of the APT in a single-agent economy. In the next section we will see that the result generalizes to an economy with many agents. The approach we use assumed that the allocation is Pareto optimal to ensure that there is a representative agent with von Neumann-Morgenstern preferences. Then a bound on the absolute risk aversion of the representative agent is derived from individual bounds.
3. Extending to many agents

To extend the bound (5) on deviations from APT pricing to the multi-agent case, we will start with the premise that the allocation is (unconstrained) Pareto optimal. There are two alternative natural motivations for this statement. According to one motivation, all assets needed to produce optimal risk sharing already exist. Alternatively, all fundamental assets (in positive net supply) satisfy the factor model (1) and there are additional financial assets (in zero net supply, e.g., options). The financial assets complete the market, so the allocation is Pareto optimal. [See Dybvig and Ingersoll (1982) for another application of this approach.]

In the K-agent economy, each agent \( k \) has a von Neumann–Morgenstern utility function \( U_k(.) \) which is increasing and concave, and has absolute risk aversion bounded above by \( A_k \). In a Pareto optimal allocation, the aggregate allocation is the same as that which would be chosen by a single agent whose vN–M preferences are given by the value of the solution to the following problem for some positive social welfare weights \( \lambda_1, \ldots, \lambda_k \):\(^8\)

\[
\text{Problem 2. Choose } w_1, w_2, \ldots, w_K \text{ to }
\max \sum_{k=1}^K \lambda_k u_k(w_k) \quad \text{subject to } \frac{1}{K} \sum_{k=1}^K w_k = w.
\]

If we let \( h_k(w) \) be the solution to Problem 2, we have that \( h_k(w) \) is characterized by the following first-order condition in which \( v(w) \) is a Lagrangian multiplier:

\[
\lambda_k u'_k(h_k(w)) = v(w), \quad (10)
\]

\[
\frac{1}{K} \sum_{k=1}^K h_k(w) = w. \quad (11)
\]

We also require the definition of the representative agent’s utility

\[^8\text{This result from welfare economics can be seen as follows. A Pareto optimum implies that we are maximizing expected utility of agent 1 subject to the reservation utility levels of the others, i.e., it solves the following problem: Choose } \tilde{w}_1, \ldots, \tilde{w}_K \text{ to }
\max \mathbb{E} u_1(\tilde{w}_1) \quad \text{subject to } \frac{1}{K} \sum_{k=1}^K \tilde{w}_k = \tilde{w}, \quad \mathbb{E} u_i(\tilde{w}_k) = \tilde{u}_i \quad \forall i \neq 1.
\]

The first-order conditions for this maximum are given by \( \mathbb{E} u'_1(\tilde{w}_1) = \mu_i \mathbb{E} u'_i(\tilde{w}_i) \forall i \). These first-order conditions are equivalent to the first-order conditions for the following problem: Choose \( \tilde{w}_1, \ldots, \tilde{w}_K \) to

\[
\max \mathbb{E} \left( \sum_{k=1}^K \lambda_k u_k(\tilde{w}_k) \right) \quad \text{subject to } \frac{1}{K} \sum_{k=1}^K \tilde{w}_k = \tilde{w}, \quad \text{for } \lambda_1 = 1, \text{ and } \lambda_i = \mu_i \text{ for } i \neq 1.
\]
function,
\[ v(w) = \sum_{k=1}^{K} \lambda_k u_k(h_k(w)). \]  
(12)

(Note that \( v \) inherits monotonicity and concavity from the \( u_k \)'s.) The following result shows how the risk aversion of the representative agent is related to the risk aversion of the individuals.

**Lemma 1.** The absolute risk aversion of the representative agent is given by
\[ A(w) = \left( \frac{1}{K} \sum_{k=1}^{K} (A_k(h_k(w)))^{-1} \right)^{-1}, \]
i.e., is the harmonic mean of the individuals' risk aversions. Therefore, an upper bound for the representative-agent's absolute risk aversion is given by
\[ \bar{A} = \left( \frac{1}{K} \sum_{k=1}^{K} \bar{A}_k^{-1} \right)^{-1}, \]
or the harmonic mean of the individual bounds.

**Proof.** Take the log of each side of (10) and differentiate, to get that for all \( k \),
\[ A_k(h_k(w))h'_k(w) = -v'(w)/v(w) = A(w), \]  
(13)

where the second equality follows from \( v'(w) = v(w) \) [since the Lagrangian multiplier \( v(w) \) in a maximization problem is the derivative of the value function \( v(w) \) — this is just the envelope theorem].

\[ 1 = \frac{1}{K} \sum_{k=1}^{K} h'_k(w) = \frac{1}{K} \sum_{k=1}^{K} A(w)/A_k(h_k(w)), \]
(by 13), or
\[ A(w) = K \left( \sum_{k=1}^{K} (A_k(h_k(w)))^{-1} \right)^{-1} = \left( \frac{1}{K} \sum_{k=1}^{K} (A_k(h_k(w)))^{-1} \right)^{-1}, \]
which is one required result. The other result is immediate, since the right-hand side of this equation is increasing in \( A_k(h_k(w)) \). Q.E.D.

9To verify this claim in this case, differentiate (12), substitute for \( \lambda_k u'(h_k(w)) \) using (10), and substitute \( (1/K) \sum_{k=1}^{K} h'_k(w) = 1 \) from (11) differentiated once.
Now we are ready to combine the results from Theorem 1 and Lemma 1 to get a result which is valid in an economy with many agents.

Theorem 2. Suppose that asset payoffs are given by the factor equation (1), that agents in the economy face problems like Problem 1 (but with $u$, $z$, and $w_0$ indexed by $k$), and that in aggregate they hold a portfolio $z_1, \ldots, z_I$. If

(a) assumption (a) of Theorem 1 holds,
(b) each agent $k$ has an increasing and strictly concave von Neumann–Morgenstern utility function $u_k$ which is three times continuously differentiable and has absolute risk aversion $A_k(x) = -u_k''(x)/u_k'(x)$ which is non-increasing and bounded above by $A_k$, and
(c) each asset is in positive net supply, i.e., $z_i > 0$ for all $i$,

then for all $i$,

$$p_i = \rho c_i + \sum_{j=1}^{J} \beta_{ij} z_j + \delta_i,$$

where $\rho$ and $\bar{z}_j$ are as in Theorem 1 except defined relative to the representative individual, and the deviation from APT pricing satisfies the following bound:

$$|\delta_i| \leq \rho e^{A z_i/K} \text{var}(\bar{e}_i) \hat{A} z_i / K,$$

where $\hat{A}$ is the harmonic mean of the agents' absolute risk aversion bounds.

Proof. We have already shown in Lemma 1 that the representative-agent's absolute risk aversion is bounded above by $\hat{A}$, so (14) and (15) will follow from Theorem 1 once we show that $v'''(w) \geq 0$. But by assumption, each agent $k$ has non-increasing absolute risk aversion. Since each $h_k(w)$ is increasing, Lemma 1 tells us that the same is true for $v$. But this implies that $v''' \geq 0$ by a standard argument. Q.E.D.

Eq. (15) tells us precisely how the approximation (5) in Theorem 1 adjusts for a multi-agent economy: the asset supplies are replaced with per capita figures and the bound on absolute risk aversion is replaced by the harmonic

\[\frac{1}{K} \sum_{k=1}^{K} (A_k)^{-1}\]

We know that

\[-(v''(w)v'(w) + (v'''(w))^2) / (v'(w))^2 \leq 0,\]

which gives the required result since $v'(w) > 0$ and $(v'(w))^2 \geq 0$. 

\[d/dw(-v''(w)/v'(w)) \leq 0 \quad \text{or} \quad -(v''(w)v'(w) - (v'''(w))^2) / (v'(w))^2 \leq 0,\]
mean of individual bounds. Therefore, all that is required for the APT to be a good approximation is that

(i) the implicit riskless discount factor $\rho$ must be reasonable,
(ii) the 'typical' risk aversion must not be too high,
(iii) the per capita asset supplies must be small, and
(iv) the idiosyncratic variance must not be too big.

All of these conditions are reasonable. Condition (ii) is even more reasonable than it might seem, since as is well known that the harmonic mean puts the most weight on small values of absolute risk aversion. Notice also that (i) requires that $\rho$ not be too large, which is fine since $\rho \leq 1$ provided interest rates are not negative. See section 5 for an estimate of the actual size of this bound.

As with Theorem 1, Theorem 2 can be restated in terms of expected return. The derivation is the same as before, so it is omitted.

Corollary 2. Under the conditions of Theorem 2, the deviations from APT pricing, in terms of expected return, are given by

$$\left| \rho^{-1} \delta_i/p_i \right| \leq e^{\alpha_i/k} \var(\tilde{e}_i/p_i) \tilde{A}_i \alpha_i/K.$$ 

This bound is the same as in Theorem 2, except that the variance is now the variance of return due to idiosyncratic noise, and the per capita supply $x_i/K$ in shares is replaced by the per capita supply $\alpha_i p_i/K$ in value (except in the exponent).

4. Relationship to sequence economies

Sequence economy results are weaker than our result of section 3 in the sense that they don't give an explicit measure of how accurate APT pricing is in a finite economy. However, it is worthwhile to illustrate how our finite economy result can be used to prove APT pricing in a sequence economy, to put our result in perspective with the previous literature.

Assume there are $J$ factors (the same for each economy in the sequence). Define assets with payoffs $x_1, x_2, \ldots$, with aggregate supplies $c_1, c_2, \ldots$, where

$$x_i = c_i + \sum_{j=1}^{J} \beta_{ij} f_j + \tilde{\epsilon}_i$$

satisfies (a) of Theorem 1, and for simplicity asset $i$ and its supply are the same in all economies in which it appears. Further assume a sequence of agents $k=1, 2, \ldots$, with utility functions satisfying (b) of Theorem 2. Let economy $n$ consist of assets $i=1, \ldots, n$ and agents $k=1, 2, \ldots, n$ (by
Theorem 2, endowments don't matter). Assuming \( A^* = \lim_{n \to \infty} \hat{A} \) and \( \rho^* = \lim_{n \to \infty} \rho \) exist and are finite, and that \( x_i \) is uniformly bounded by \( \bar{x} \) across \( i \), it follows from Theorem 2 that the largest deviation from APT pricing in the \( n \)th economy tends to 0:

\[
\lim_{n \to \infty} \left( \sup_{i=1, \ldots, n} |\delta_i^*| \right) \leq \lim_{n \to \infty} \left( \sup_{i=1, \ldots, n} \left( \rho e^{A^*/K} \text{var}(\hat{r}_i) \hat{A} x_i/K \right) \right)
\leq \lim_{n \to \infty} (\rho \hat{A} e^{A^*/K} \bar{x} / n = \rho^* A^* \lim e^{A^*/n} \bar{x} / n = 0.
\]

This quick informal derivation illustrates that it is possible to do sequence economies with less extreme assumptions and more direct arguments than in previous papers.

5. A back-of-the-envelope computation of the bound

This section attempts to get a quick and coarse estimate of the bound indicated in (15). Since interest rates are positive, \( \rho < 1 \). Absolute risk aversion equals relative risk divided by the wealth level. A large value for relative risk aversion from empirical tests is 5 [see Brown and Gibbons (1982) for a survey of the empirical estimates]. A conservative wealth level for a typical investor is $100,000. (Keep in mind that the harmonic mean puts much greater weight on less risk averse agents — wealthier agents under CRRA.) A very large own standard derivation for monthly data might be $20/share. (Of course this number is even more generous for weekly or daily data, since the own standard deviation is smaller for smaller time intervals.) Finally, a large per capita stock supply is 5 shares. Therefore,

\[
|\delta_i| \leq 1 \cdot e^{5/100,000} \cdot 5 \cdot 20^2 \cdot 100,000 = 0.1 \text{ or } 10\,\text{c.}
\]

This value is small, but in fact is much too large for any given security, since at each point we have made an extreme assumption. For example, only the largest firms have anything near to 5 shares per capita outstanding, and these firms do not have standard deviations of $20 in their prices. Also, $100,000 is a very conservative estimate of the wealth of the agent who is marginal in terms of asset pricing. Consequently, while 10¢ a share is a small deviation in share price, this is probably much too large, since it is based on several 'worst cases' holding at once. Therefore, we can conclude that as a practical matter, APT pricing is a very good approximation and the deviations from APT pricing can safely be ignored, in terms of pricing.

To estimate the bound in terms of expected return, using Corollary 2, we will first notice that the variance of \( \hat{r}_i \) and the mean of \( \hat{r}_i - 1 \) should both be proportional to the length of the time interval, by an intuition based on the diffusion model or the law of large numbers. Therefore, if returns and \( \hat{r}_i \) are
both put on an annualized basis, the bound in Corollary 1 should not depend on the time interval (e.g., annual, monthly, or daily). Therefore, for convenience, assume annual returns. Then 20% is large for the standard deviation of \( \delta_i / p_t \) — stocks with larger values will tend to have offsetting very small \( x_i p_t \). A very large value of \( x_i p_t \) is $200 per capita. Take \( A \) and \( \bar{A} x_i \) (for the exponent) as above. Then the bound on returns is

\[
|\rho^{-1} \delta_i / p_t| \leq e^{(5/100,000)^5} \cdot 20^2 (5/100,000) 200 = 0.0004 = 0.04\% 
\]

or four hundredths of one percent, on an annualized basis. Again this is small enough to be ignored, and the estimate is conservative in the sense that we have simultaneously assumed several worst cases.

6. Conclusion

We have given a simple yet rigorous derivation of Ross’s APT, with an explicit bound on the deviation from APT prices in a finite economy. A rough computation has indicated that the deviations from APT pricing are negligible in our economy. Future theoretical research can add “jazzy” features (e.g., correlated error terms), derive conditions for factor prices to be non-zero, derive conditions for the factors to all be independently significant, and introduce asymmetric information. On the empirical side, there remains the continuing challenge of testing the model and identifying the factors, and the challenge of discovering the time series properties of the factor loadings.\(^{11}\)

\(^{11}\)The interface between theory and empirical work on the APT has been subject to much confusion. See Dybvig and Ross (1983) which addresses these issues.

References

Brown, David and Michael Gibbons, 1982, A simple econometric approach for utility-based asset pricing models (Graduate School of Business, Stanford University, Stanford, CA).


Dybvig, Philip and Stephen Ross, 1983, Yes, the APT is testable, Journal of Finance, forthcoming.

Ingersoll, Jonathan, Jr., 1981, Some results in the theory of arbitrage pricing, Working paper (School of Management, Yale University, New Haven, CT).
Ross, Stephen, 1982, Intertemporal asset pricing theories, Working paper (School of Management, Yale University, New Haven, CT).