A Simple Approach to Arbitrage Pricing Theory

GUR HUBERMAN*

Graduate School of Business, University of Chicago,
Chicago, Illinois 60637

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1. INTRODUCTION

The arbitrage theory of capital asset pricing was developed by Ross [9, 10, 11] as an alternative to the mean-variance capital asset pricing model (CAPM), whose main conclusion is that the market portfolio is mean-variance efficient. Its formal statement entails the following notation. A given asset $i$ has mean return $E_i$ and the market portfolio has mean return $E_m$ and variance $\sigma_m^2$. The covariance between the return on asset $i$ and the return on the market portfolio is $\sigma_{im}$, and the riskless interest rate is $r$. The CAPM asserts that

$$E_i = r + \lambda b_i,$$  \hspace{1cm} (1.1)

where

$$\lambda = E_m - r,$$

and

$$b_i = \sigma_{im} / \sigma_m^2$$ \hspace{1cm} (1.2)

is the "beta coefficient" of asset $i$.

Normality of the returns of the capital assets or quadratic preferences of their holders are the assumptions which lead to (1.1)–(1.2). Theoretically and empirically it is difficult to justify the assumptions of the CAPM. Moreover, the CAPM has been under strong criticism because of its dubious empirical content (cf. [7]). The market portfolio is practically not obs}

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vable, and a statement on the market portfolio (such as the CAPM) is difficult to test empirically. Yet the linear relation (1.1) is appealing in its simplicity and in its ready interpretations. The arbitrage pricing theory [10, 11] is an alternative theory to mean-variance theories, an alternative which implies an approximately linear relation like (1.1). In [10] Ross elaborated on the economic interpretation of the arbitrage pricing theory and its relation to other models, whereas in [11] he provided a rigorous treatment of the theory. Recent interest in the APT is evident from papers elaborating on the theory (e.g., Chamberlin and Rothschild [1], Connor [4] and Kwon [5, 6]) as well as on its empirical aspects (e.g., Chen [2, 3] and Roll and Ross [8]).

The main advantage of Ross' arbitrage pricing theory is that its empirical testability does not hinge upon knowledge of the market's portfolio. Unfortunately, Ross' analysis is difficult to follow. He does not provide an explicit definition of arbitrage and his proof—unlike the intuitively appealing introductory remarks in [11]—involves assumptions on agents' preferences as well as "no arbitrage" assumptions.

Here arbitrage is defined and the intuition is formalized to obtain a simple proof that no arbitrage implies Ross' linear-like relation among mean returns and covariances. The main lines of the proof are illustrated in the following paragraphs.

Consider an economy with n risky assets whose returns are denoted by \( \bar{x}_i \) \((i = 1, ..., n)\) and they are generated by a factor model

\[
\bar{x}_i = E_i + \beta_i \delta_i + \epsilon_i \quad (i = 1, ..., n),
\]

(1.3)

where the expectations \( E \delta = E\epsilon = 0 \) \((t = 1, ..., n)\), the \( \epsilon_i \) are uncorrelated and their variances are bounded. Relying on results from linear algebra, express the vector \( E \) (whose ith component is \( E_i \)) as a linear combination of the vector \( e \) (whose ith component is 1), the vector \( \beta \) (whose ith component is \( \beta_i \)) and a third vector \( c \) which is orthogonal both to \( e \) and to \( \beta \). In other words, one can always find a vector \( c \) such that

\[
E = \rho e + \gamma \beta + c,
\]

(1.4)

where \( \rho \) and \( \gamma \) are scalars,

\[
e c \equiv \sum_{i=1}^{n} c_i = 0,
\]

(1.5)

and

\[
\beta c \equiv \sum_{i=1}^{n} \beta_i c_i = 0.
\]

(1.6)

\footnote{Gregory Connor used this idea in an earlier work of his [4].}
Next, consider a portfolio which is proportional to \( c \), namely \( \alpha c \) (\( \alpha \) is a scalar). Note that it costs nothing to acquire such a portfolio because its components (the dollar amount put into each asset) sum to zero by (1.5). We shall call such a portfolio an arbitrage portfolio. Also, by (1.6) this is a zero-beta portfolio. The return on this portfolio is

\[
\alpha \bar{c} = \alpha \sum_{i=1}^{n} \bar{x}_i c_i = \alpha \sum_{i=1}^{n} c_i^2 + \alpha \sum_{i=1}^{n} c_i \tilde{\epsilon}_i, \tag{1.7}
\]

by virtue of the decomposition (1.4) and the orthogonality relations (1.5) and (1.6). It is important to notice that the expected return on the portfolio \( \alpha c \) is proportional to \( \alpha \) (and \( \sum_{i=1}^{n} c_i^2 \)), whereas an upper bound on the variance of its return is proportional to \( \alpha^2 \) (and \( \sum_{i=1}^{n} c_i^2 \)).

Suppose now that the number of assets \( n \) increases to infinity. Think of arbitrage in this environment as the opportunity to create a sequence of arbitrage portfolios whose expected returns increase to infinity while the variances of their returns decrease to zero. If the sum \( \sum_{i=1}^{n} c_i^2 \) increased to infinity as \( n \) did, then one could find such arbitrage opportunities as follows. Set \( \alpha = \frac{1}{(\sum_{i=1}^{n} c_i^2)^{3/4}} \) and use the portfolio \( \alpha c \). The reason why such a choice of \( \alpha \) will create the arbitrage is that the expected return on the portfolio is proportional to \( \alpha \) (and with \( \alpha = \frac{1}{(\sum_{i=1}^{n} c_i^2)^{3/4}} \) it equals \( \alpha \sum_{i=1}^{n} c_i^2 = (\sum_{i=1}^{n} c_i^2)^{1/4} \)), while its variance is proportional to \( \alpha^2 \) (and with \( \alpha = \frac{1}{(\sum_{i=1}^{n} c_i^2)^{3/4}} \) it equals \( \alpha^2 \sum_{i=1}^{n} c_i^2 = 1/(\sum_{i=1}^{n} c_i^2)^{1/2} \)).

Therefore, if there are no arbitrage opportunities (as described above) the sum \( \sum_{i=1}^{n} c_i^2 \) cannot increase to infinity as \( n \) does. In particular, when the number of assets \( n \) is large, most of the \( c_i \)'s are small and approximately zero. Going back to the original decomposition (1.4) we conclude that \( E_i \approx \rho + \gamma \beta_i \) for most of the assets.

When motivating his proof, Ross [11, p. 342] emphasized the role of "well-diversified" arbitrage portfolios. He indicated that the law of large numbers was the driving force behind the diminishing contribution of the idiosyncratic risks \( \tilde{\epsilon}_i \) to the overall risks of the arbitrage portfolios. The portfolios presented above, \( \alpha c \), need not be well diversified, but they satisfy the orthogonality conditions (1.5) and (1.6). It is the judicious choice of the scalar \( \alpha \) that enables us to apply an idea, which is in the spirit of the proof of the law of large numbers.

Section 2 of this paper presents the formal model, a precise statement of the result and a rigorous proof. In the closing section an attempt is made to interpret the linear-like pricing relation and to justify the no-arbitrage assumption in an equilibrated economy of von Neumann–Morgenstern expected utility maximizers.
2. Arbitrage Pricing

The arbitrage pricing theory considers a sequence of economies with increasing sets of risky assets. In the \( n \)th economy there are \( n \) risky assets whose returns are generated by a \( k \)-factor model (\( k \) is a fixed number). Loosely speaking, arbitrage is the possibility to have arbitrarily large returns as the number of available assets grows. We will show that in the absence of arbitrage a relation like (1.1) holds, namely (2.9).

Formally, in the \( n \)th economy, we consider an array of returns on risky assets \( \{ x^n_i : i = 1, \ldots, n \} \). These returns are generated by a \( k \)-factor linear model of the form

\[
\tilde{x}_i^n = E^n_i + \beta^n_{i1} \tilde{\delta}_1^n + \beta^n_{i2} \tilde{\delta}_2^n + \cdots + \beta^n_{ik} \tilde{\delta}_k^n + \varepsilon_i^n \quad (i = 1, 2, \ldots, n),
\]

where

\[
E \tilde{\delta}_j^n = 0 \quad (j = 1, \ldots, k), \quad F \tilde{\varepsilon}_i^n = 0 \quad (i = 1, \ldots, n),
\]

\[
E \tilde{\varepsilon}_i^n \tilde{\varepsilon}_j^n = 0 \quad \text{if} \quad i \neq j,
\]

and \( \text{Var} \ \tilde{\varepsilon}_i^n \leq \tilde{\sigma}^2 \quad (i = 1, \ldots, n) \),

where \( \tilde{\sigma}^2 \) is a fixed (positive) number. Using standard matrix notation we can rewrite (2.1) as

\[
\tilde{x}^n = E^n + \beta^n \tilde{\delta}^n + \varepsilon^n,
\]

where \( \beta^n \) is the \( n \times k \) matrix whose elements are \( \beta^n_{ij} \) (\( i = 1, \ldots, n; j = 1, \ldots, k \)).

A portfolio \( c^n \in \mathbb{R}^n \) in the \( n \)th economy is an arbitrage portfolio if \( c^n e^n = 0 \), where \( e^n = (1, 1, \ldots, 1) \in \mathbb{R}^n \). The return on a portfolio \( c \) is

\[
\tilde{z}(c) = c \tilde{x}^n = c E^n + c \beta^n \tilde{\delta}^n + c \varepsilon^n.
\]

Arbitrage is the existence of a subsequence \( n' \) of arbitrage portfolios whose returns \( \tilde{z}(c^n') \) satisfy

\[
\lim_{n' \to \infty} E \tilde{z}(c^n') = +\infty,
\]

and

\[
\lim_{n' \to \infty} \text{Var} \ \tilde{z}(c^n') = 0.
\]

In Section 3 we relate (2.7)–(2.8) to standard probabilistic convergence concepts, and discuss how von Neumann–Morgenstern expected utility maximizers view (2.7)–(2.8).
In Theorem 1 we show that the absence of arbitrage implies an approximation to a linear relation like (1.1).

**Theorem 1.** Suppose the returns on the risky investments satisfy (2.1)–(2.4) and there is no arbitrage. Then for \( n = 1, 2, \ldots \), there exists \( \rho^n, \gamma^n_1, \ldots, \gamma^n_k \), and an \( A \) such that

\[
\sum_{i=1}^{n} \left( E_i^n - \rho^n - \sum_{j=1}^{k} \beta^n_{ij} \gamma^n_j \right)^2 \leq A, \quad \text{for } n = 1, 2, \ldots \tag{2.9}
\]

**Proof.** Using the orthogonal projection of \( E^n \) into the linear subspace spanned by \( e^n \) and the columns of \( \beta^n \), one obtains the representation

\[
E^n = \rho^n e + \beta^n \gamma^n + c^n, \tag{2.10}
\]

where

\[
\gamma^n \in R^k, \quad e^n c^n = 0, \tag{2.11}
\]

and

\[
\beta^n c^n = 0. \tag{2.12}
\]

Note that \( \|c^n\|^2 \equiv \sum_{i=1}^{n} (c_i^n)^2 = \sum_{i=1}^{n} \left( E_i^n - \rho^n - \sum_{j=1}^{k} \gamma^n_j \beta^n_{ij} \right)^2 \), and assume that the result is false. Consequently, there is an increasing subsequence \( (n') \) with

\[
\lim_{n' \to \infty} \|c^{n'}\| = +\infty \tag{2.13}
\]

Let \( p \) be fixed between \(-1\) and \(-1/2\), and consider the portfolio \( d^{n'} = \alpha_{n'} c^{n'} \), where

\[
\alpha_{n'} = \|c^{n'}\|^{2p}. \tag{2.14}
\]

By (2.11), \( d^{n'} \) is an arbitrage portfolio for each \( n' \). Use (2.10)–(2.12) to see that its return

\[
\bar{z}(d^{n'}) = \alpha_{n'} \|c^{n'}\|^2 + \alpha_{n'} c^{n'} \bar{c}^{n'}. \tag{2.15}
\]

Note that

\[
E \bar{z}(d^{n'}) = \alpha_{n'} \|c^{n'}\|^2 = \|c^{n'}\|^2 + 2p, \tag{2.16}
\]

so (by (2.13)–(2.14)),

\[
\lim_{n' \to \infty} E \bar{z}(d^{n'}) = +\infty. \tag{2.17}
\]
On the other hand (using (2.3), (2.4))

$$\text{Var } \tilde{z}(d^n) \leq \sigma^2 \alpha_n^2 \|c^n\|^2 = \sigma^2 \|c^n\|^2 + 4^p,$$

(2.18)

so (by (2.13)),

$$\lim_{n' \to \infty} \text{Var } \tilde{z}(d^n') = 0,$$

thus completing the proof. 

Next, consider a stationary model, in which $E_i^n = E_i$ and $\beta^n_{ij} = \beta_{ij}$ for all $i, j$ and $n$. In other words, (2.5) is replaced by

$$\tilde{x}^n = E + \beta \delta^n + \tilde{e}^n.$$

(2.5')

The stationary model is the one considered originally by Ross [11]. The nonstationary model is more general than the stationary model but its result (2.9) is not as elegantly presentable as the result in the stationary case (2.9').

**Theorem 2.** Suppose the returns on the risky investments satisfy (2.5'), (2.2)-(2.4), and there is no arbitrage. Then there exist $\rho, \gamma_1, ..., \gamma_k$ such that

$$\sum_{i=1}^{\infty} \left( E_i - \rho - \sum_{j=1}^{k} \beta_{ij} \gamma_j \right)^2 < \infty.$$

(2.9')

**Proof.** Consider the $n \times (k+1)$ matrix $B^n$ whose $(i, j)$ entry is 1 if $j = 1$ and $\beta^n_{ij-1}$ if $1 < j \leq k + 1$. Let $r(n)$ be the rank of $B^n$. Since $1 \leq r(n) \leq r(n+1) \leq k + 1$ for all $n$, and $r(n)$ is an integer, there is an $\bar{n}$ such that $r(n) = r(\bar{n})$ for all $n \geq \bar{n}$. Let $n \geq \bar{n}$ be fixed. By permuting the columns of $B^n$ we may assume that its first $r(\bar{n})$ columns can be expressed as linear combinations of the first $r(\bar{n})$ columns. Define the set $H^n$ by

$$H^n = \left\{ (\rho, \gamma_1, ..., \gamma_k) : \sum_{i=1}^{n} \left( E_i - \rho - \sum_{j=1}^{k} \beta_{ij} \gamma_j \right)^2 \leq A, \gamma_j = 0, \right\}$$

$$\text{for } r(\bar{n}) < j \leq k \}.$$

where $A$ is the $A$ whose existence was asserted in Theorem 1. Note that $H^n$ is nonempty (by Theorem 1), compact for $n \geq \bar{n}$ and $H^n \subset H^{n+1}$. Therefore, \( \bigcap_{n=1}^{\infty} H^n \) is nonempty. Since every $k + 1$ tuple $(\rho, \gamma_1, ..., \gamma_k) \in \bigcap_{n=1}^{\infty} H^n$ satisfies (2.9'), the proof is complete.

Finally, we turn attention to the case where a risk free asset exists, i.e., where there is an additional asset in the $n$th economy, whose return, say, $x^n_0$, satisfies

$$x^n_0 = r^n_0.$$

(2.19)
Now look at excess returns of the risky assets (excess relative to the riskless rate), i.e., at
\[
\tilde{y}_i^n = \tilde{x}_i^n - r_0^n, \quad i = 1, 2, \ldots, n.
\]

Note that any arbitrage portfolio \((c_0, c_1, \ldots, c_n) \in \mathbb{R}^{n+1}\) of \(x_0^n, \tilde{x}_1^n, \ldots, \tilde{x}_n^n\) (which of course satisfies \(\sum_{i=0}^n c_i = 0\)) is equivalent to a vector \((c_1, \ldots, c_n) \in \mathbb{R}^n\) indicating a wealth allocation among the risky assets. Using this idea one can go through the same analysis as in Theorem 1 with the excess returns vector \(\tilde{y}_n\), the decomposition (2.10) replaced by
\[
E^n - r_0^n e = \beta^n \gamma^n + c^n,
\]
and (2.11) deleted.

Consequently, one has

**Corollary.** Suppose the returns on the risky investments satisfy (2.1)-(2.4), there is a risk free asset satisfying (2.19) and there is no arbitrage. Then there exist \(\gamma_1^n, \gamma_2^n, \ldots, \gamma_k^n\) such that
\[
\sum_{i=1}^n \left( E_i^n - r_0^n - \sum_{j=1}^k \beta_{ij} \gamma_j^n \right)^2 \leq A \quad \text{for } n = 1, 2, \ldots. \tag{2.20}
\]

**Remark.** Analogously, a similar result holds for the stationary model.

### 3. Discussion

The interpretation of (2.9) or (2.9') is straightforward: for most of the assets in a large economy, the mean return on an asset is approximately linearly related to the covariances of the asset's returns with economy-wide common factors. As the number of assets becomes large, the linear approximation improves and most of the assets' mean returns are almost exact linear functions of the appropriate covariances.

Next, consider the probabilistic implications of arbitrage returns satisfying (2.7)-(2.8). Given a sequence of random returns \(\tilde{z}(c^n)\) which satisfy (2.7)-(2.8), we can apply Chebychev's inequality to see that along this sequence \(\lim_{n \to \infty} \tilde{z}(c^n) = +\infty\) in probability (i.e., for all \(M > 0\), \(\lim_{n \to \infty} \Pr\{\tilde{z}(c^n) \geq M\} = 1\)). Furthermore, along a subsequence \(n^*\), a stronger convergence holds: \(\lim_{n^* \to \infty} \tilde{z}(c^d) = +\infty\) almost surely (i.e., for all \(M > 0\), \(\Pr\{\lim \inf_{n^* \to \infty} \tilde{z}(c^n) \geq M\} = 1\)).

Are arbitrage portfolio which satisfy (2.7)-(2.8) desirable for an expected utility maximizer? In other words, do (2.7)-(2.8) suffice to assert that \(\lim_{n \to \infty} EU(\tilde{z}(c^n)) = U(+\infty)\) for any monotone concave utility function \(U\)? The negative answer is illustrated by the following examples.
The first example considers a utility function which is \(-\infty\) for nonpositive wealth levels, whereas the second example is for an exponential utility which takes finite values for finite wealth levels.

1. The returns \(\tilde{z}(c^n)\) are 0, \(n\) and \(2n\) with probabilities \(1/n^3\), \(1 - 2/n^3\), and \(1/n^3\), respectively. The utility function \(U(x) = -1/x\) for \(x > 0\) and \(U(x) = -\infty\) for \(x \leq 0\). Then \(EU(\tilde{z}(c^n)) = -\infty\) although \(\tilde{z}(c^n)\) satisfies (2.7)–(2.8).

2. The returns \(\tilde{z}(c^n)\) are \(-n\), \(n\), and \(3n\) with probabilities \(1/n^3\), \(1 - 2/n^3\), and \(1/n^3\), respectively. The utility function is \(U(x) = -\exp(-x)\). Then \(EU(\tilde{z}(c^n)) \leq -n^3 \exp(n)\), so \(\lim_{n \to \infty} EU(\tilde{z}(c^n)) = -\infty\), although (2.7)–(2.8) are met.

General conditions which assert that (2.7)–(2.8) imply \(\lim_{n \to \infty} EU(\tilde{z}(c^n)) = U(+\infty)\) are not known. As shown in [11, Appendix 2], utility functions which are bounded below or uniformly integrable utility functions will possess this property.

We conclude that one needs to make assumptions on agents' preferences in order to relate existence of equilibria to absence of arbitrage. This task is beyond the scope of this paper. However, it is straightforward to see that if the economies satisfy the assumptions made by Ross (see [11], especially the first paragraph on p. 349 and Appendix 2), then no arbitrage can exist. In fact, a result of the type "no arbitrage implies a certain behavior of returns," should involve no consideration of the preference structure of the agents involved. Our analysis is in this spirit, because it involves no assumptions on utilities. Other than the simple proof, this may be another contribution of this work.

References

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